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Dynamic risk in the IS-LM model with delayed

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In this paper we present a stability analysis for a IS-LM model (Investment Saving-Liquidity preference Money supply) and according to Kalecki we considered the idea of a time delay in the model. We establish conditions to prove that the delay gain or lose stability and a Hopf bifurcation occurs. For the stability analysis we consider the particular case where the investment (I) and the demand for money (L) are nonlinear.

Delay, IS-LM model, Hopf Bifurcation.

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Introduction

Investment is a topic of great importance in the economy. The business cycle models of Kaldor and Kalecki use an investment function which is based more on the principle of profit and principle of acceleration. In the model of Kaldor, gross investment depends on the level of performance and capital stock. For a given amount of real capital, investment depends on the level of profit, which depends on the activity level. Kaldor's assumptions present a nonlinear investment and saving functions and their change over time leading to a cycle. On this model we have studied the possibility of possessing a unique limit cycle, a major breakthrough came when necessary and sufficient conditions for the existence of a limit cycle in the model is established, has also studied the coexistence of a limit cycle and balance.

The macroeconomic business cycle Kaldor [1] of the form

$$\frac{dY}{dt} = \alpha[I(Y,K) - S(Y,K)]$$
$$\frac{dK}{dt} = I(Y,K) - \delta K, \qquad (1)$$

Where I is the investment, S saving function, and the gross domestic product, K the capital stock, α the coefficient of market adjustment and δ is the depreciation rate of capital stock. Kaldor assumes that the investment function I is not linear in Y and has an "s". As an example of this type of function we can mention the proposed Bischi et al [2] for your model, this function is:

$$I(t) = \sigma\mu + \gamma\left(\frac{\sigma\mu}{\delta} - K(t)\right) + \arctan(Y(t) - \mu), \qquad (2)$$

Where $\sigma\mu/\delta$ is the normal level of capital stock. And consider two short investment components, the first is proportional to the difference between the stock of capital and the current average stock, according to a coefficient $\gamma > 0$, usually explained by the presence of adjustment costs; the second is an increasing function of the difference between "normal" income and level, indicated by μ. This function has an "s" seen as a function that depends on income$Y$.

The model Kalecki business cycle [3] was proposed a few years earlier than that of Kaldor. Kalecki assumed that the portion of the gain is invested savings and capital grows because of past investment decisions. There is a gestation period or time delay, after which the assets are available for production.

In Kalecki's theory of the fundamental role is played by the time delay T related to investment decisions. He distinguishes three stages of investment: investment orders I, production of capital goods A, and capital goods ended D. The change in capital stock is due to investment orders in the past,

$$\frac{dK(t)}{dt} = D(t) - U = I(t-T) - U, \qquad (3)$$

Where U denotes the capital depreciation. Krawiec and Szydlowski [4] formulated the business cycle model Kaldor-Kalecki based on dynamic multiplier which is the basis of both approaches, following the approach of Kalecki investment and the time delay between investment decisions and implementation , obtain the system of differential equations with delay:

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$$\frac{dY}{dt} = \alpha\big[I\big(Y(t), K(t)\big) - S\big(Y(t), K(t)\big)\big]$$
$$\frac{dK}{dt} = I\big(Y(t-T), K(t)\big) - \delta K(t), \qquad (4)$$

Where T = constant is the time delay. Investment depends on the gain at the time investment decisions are made and the stock of capital investment when the time ends. The latter is due to the fact that at time tT, there are some investments that will end between tT and t. It is assumed that the capital stock produced in this period is taken into account when new investments are planned.

The authors conclude that the reconstructed model of the business cycle based on the Kaldor model parameter and time delay associated with Kalecki investment decisions generates limit cycles in phase space. The crucial role in the creation of the limit cycle is the time delay, rather than the assumption of the role of investment in the form of "s".

Torre [5] reviewed and updated the model by replacing the stock of capital with the interest rate R (t) to formulate the next business cycle model standard IS-LM:

$$\frac{dY}{dt} = \alpha\big[I\big(Y(t), R(t)\big) - S\big(Y(t), R(t)\big)\big]$$
$$\frac{dR}{dt} = \beta\big[L\big(Y(t), R(t)\big) - M\big], \qquad (5)$$

Where M is the constant flow of money, β is the coefficient of adjustment in the money market and L is the demand for money. Apparently Torre who was originally introduced to the economy Hopf bifurcation theorem [6] as a tool to establish the existence of closed orbits in dynamical systems. As Kaldor model combination and Torre Gabisch and Lorenz [7] considered an IS-LM model amended as follows:

$$\frac{dY}{dt} = \alpha\big[I\big(Y(t), K(t), R(t)\big) - S\big(Y(t), R(t)\big)\big]$$
$$\frac{dK}{dt} = I\big(Y(t), K(t), R(t)\big) - \delta K(t) \qquad (6)$$
$$\frac{dR}{dt} = \beta\big[L\big(Y(t), R(t)\big) - M\big],$$

Where δ is the depreciation rate of capital stock. Based on the idea of Kalecki delay in time, Cai [8] presented the following IS-LM model with delay:

$$\frac{dY}{dt} = \alpha\big[I\big(Y(t), K(t), R(t)\big) - S\big(Y(t), R(t)\big)\big]$$
$$\frac{dK}{dt} = I\big(Y(t-\tau), K(t), R(t)\big) - \delta K(t) \qquad (7)$$
$$\frac{dR}{dt} = \beta\big[L\big(Y(t), R(t)\big) - M\big].$$

In this model τ is the time delay required for the new capital is installed, Cai gave results on local stability and Hopf bifurcation.

Kaddar and Talibi [9] considered a model where the τ delay is introduced in the gross domestic product, capital stock and interest rate, arguing that the change in the capital stock is due to past investment decisions, obtaining the following model:

$$\frac{dY}{dt} = \alpha\big[I\big(Y(t), K(t), R(t)\big) - S\big(Y(t), R(t)\big)\big]$$
$$\frac{dK}{dt} = I\big(Y(t-\tau), K(t-\tau), R(t-\tau)\big) - \delta K(t) \qquad (8)$$
$$\frac{dR}{dt} = \beta\big[L\big(Y(t), R(t)\big) - M\big].$$

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Talibi and Kaddar study the dynamics of the model locally and describe the Hopf bifurcation. This assuming the functions of investment I, the saving function S, and the demand for money L as:

$$I(Y,K,R) = \eta Y - \delta_1 K - \beta_1 R$$
$$S(Y,R) = l_1 Y + \beta_2 R$$
$$L(Y,R) = l_2 Y - \beta_3 R,$$

Where $\delta_1, l_1, l_2, \beta_1, \beta_2, \beta_3$ are positive constants

According to Rocsoreanu [10], economic conditions lead to the following limitations on economic variables:

$$\begin{aligned} &\frac{\partial I}{\partial Y} > 0, \quad \frac{\partial I}{\partial R} < 0, \quad \frac{\partial I}{\partial K} < 0, \\ &\frac{\partial S}{\partial Y} > 0, \quad \frac{\partial S}{\partial R} > 0, \\ &\frac{\partial L}{\partial Y} > 0, \quad \frac{\partial L}{\partial R} < 0. \end{aligned} \tag{9}$$

Based on the above functions take investment, savings and money demand as follows, these functions being considered according to the idea of Rocsoreanu [10] and De Cesare [11]:

$$\begin{aligned} &I(Y,K,R) = A\frac{Y^a}{R^b} - cK \\ &S(Y) = sY \\ &L(Y,R) = \gamma Y + \frac{h}{R - \hat{R}} \ , \end{aligned} \tag{10}$$

Which satisfy the following constants $a, b, A, \gamma, h > 0, \ 0 < s < 1$ y $\hat{R} > 0$ es una tasa very small fixed interest generating a liquidity trap when R (t) tends to R, ie, $\frac{h}{R-\hat{R}} \to +\infty$ when $R(t) \to \hat{R}$.

Note that these functions satisfy the constraints given in (9) and also the value that is assigned to the exponent allows us to control the shape of the graph of I as a function of Y (Concave for a <1, convex for a> 1 and linear for a = 1).; the exponent b allow us to control the convexity of the graph of R as a function of R; L allows the existence of a liquidity trap and finally S is proposed as a linear function of Y as is often done in the literature.

Considering the system (8) and functions as described in (10) to analyze the system is:

$$\begin{aligned} &\frac{dY}{dt} = \alpha\left(A\frac{Y^a(t)}{R^b(t)} - cK(t) - sY(t)\right) \\ &\frac{dK}{dt} = A\frac{Y^a(t-\tau)}{R^b(t-\tau)} - cK(t-\tau) - \delta K(t) \\ &\frac{dR}{dt} = \beta\left(\gamma Y(t) + \frac{h}{R(t) - \hat{R}} - M\right). \end{aligned} \tag{11}$$

Stability analysis

In this section we perform a stability analysis of the system (11) by calculating its characteristic equation at an equilibrium point.

The equilibrium point (Y ^ *, K ^ *, R *) for the above system is given by the constant solutions of the system, note that if the solutions are constant the right side of each equation of the system (11) is equals zero and $Y(t) = Y(t-\tau) = Y^*$ for all t, the same is satisfied for K ^ *, R *, then the equilibrium point is obtained by solving the following system:

$$\begin{aligned} &0 = A\frac{(Y^*)^a}{(R^*)^b} - cK^* - sY^* \\ &0 = A\frac{(Y^*)^a}{(R^*)^b} - cK^* - \delta K^* \end{aligned} \tag{12}$$

$$M = \gamma Y^* - \frac{h}{R^* - \hat{R}}.$$

For linearization consider the system as:

$$\begin{aligned} \frac{dY}{dt} &= f_1(Y,K,R,Y_\tau,K_\tau,R_\tau) \\ \frac{dK}{dt} &= f_2(Y,K,R,Y_\tau,K_\tau,R_\tau) \\ \frac{dR}{dt} &= f_3(Y,K,R,Y_\tau,K_\tau,R_\tau). \end{aligned} \quad (13)$$

The functions f_i, $i = 1,2,3$ that depends on Y, K, R, $Y_\tau = Y(t-\tau)$,$K_\tau = K(t-\tau)$ and $R_\tau = R(t-\tau)$, are the right side of the system (11) associated with $\partial Y/\partial t$, $\partial K/\partial t$ y $\partial R/\partial t$ respectively. The linearization is given by:

$$\dot{x}(t) = Jx(t) + J_D x(t-\tau), \quad (14)$$

Con $x(t) = (x_1(t), x_2(t), x_3(t))$ donde J y J_D son jacobianos de (13) evaluados en el equilibrio (Y^*, K^*, R^*), que están dados por:

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial Y} & \frac{\partial f_1}{\partial K} & \frac{\partial f_1}{\partial R} \\ \frac{\partial f_2}{\partial Y} & \frac{\partial f_2}{\partial K} & \frac{\partial f_2}{\partial R} \\ \frac{\partial f_3}{\partial Y} & \frac{\partial f_3}{\partial K} & \frac{\partial f_3}{\partial R} \end{pmatrix} = \begin{pmatrix} \alpha\left(\frac{Aa(Y^*)^{a-1}}{(R^*)^b} - s\right) & -\alpha c & -\alpha Ab\frac{(Y^*)^a}{(R^*)^{b+1}} \\ 0 & -\delta & 0 \\ \beta\gamma & 0 & -\frac{\beta h}{\left(R^* - \hat{R}\right)^2} \end{pmatrix}$$

$$J_D = \begin{pmatrix} \frac{\partial f_1}{\partial Y_\tau} & \frac{\partial f_1}{\partial K_\tau} & \frac{\partial f_1}{\partial R_\tau} \\ \frac{\partial f_2}{\partial Y_\tau} & \frac{\partial f_2}{\partial K_\tau} & \frac{\partial f_2}{\partial R_\tau} \\ \frac{\partial f_3}{\partial Y_\tau} & \frac{\partial f_3}{\partial K_\tau} & \frac{\partial f_3}{\partial R_\tau} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ Aa\frac{(Y^*)^{a-1}}{(R^*)^b} & -c & -Ab\frac{(Y^*)^a}{(R^*)^{b+1}} \\ 0 & 0 & 0 \end{pmatrix}$$

To find the characteristic equation of the system, we seek solutions of the form $\left(c_1 e^{\lambda t}, c_2 e^{\lambda t}, c_3 e^{\lambda t}\right)$ for the linear system (14) are non-trivial. The condition for the above to happen is that the determinant of the system matrix equaled zero $\Delta(\lambda,\tau) = \lambda I - J - e^{-\lambda\tau} J_D$ (Here I is the identity matrix 3x3), Solutions obtained by considering the previously mentioned manner, is equal to zero. In our case

$$\Delta(\lambda,\tau) \coloneqq \lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 + (b_2\lambda^2 + b_1\lambda + b_0)e^{-\lambda\tau} = 0, \quad (15)$$

Where:

$$a_2 = as + \delta + a\alpha A\frac{(Y^*)^{a-1}}{(R^*)^b}$$

$$a_1 = \alpha s\delta + a\alpha A\delta\frac{(Y^*)^{a-1}}{(R^*)^b} + Ab\alpha\beta\gamma\frac{(Y^*)^a}{(R^*)^{b+1}} + \frac{\beta h}{\left(R^* - \hat{R}\right)^2}\left(\alpha s + \delta - a\alpha A\frac{(Y^*)^{a-1}}{(R^*)^b}\right)$$

$$a_0 = Ab\alpha\beta\gamma\delta\frac{(Y^*)^a}{(R^*)^{b+1}} + \frac{\alpha\beta\delta h}{\left(R^* - \hat{R}\right)^2}\left(s - aA\frac{(Y^*)^{a-1}}{(R^*)^b}\right)$$

$$b_2 = c$$

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$$b_1 = \alpha sc + \frac{c\beta h}{\left(R^* - \hat{R}\right)^2}$$
$$b_0 = \frac{\alpha sc\beta h}{\left(R^* - \hat{R}\right)^2}$$

Based on the characteristic equation we can establish the stability of equilibrium whereas an equilibrium solution is stable if all λ roots of the characteristic equation (15) are in the left half of the complex plane, this is the real part of λ, R (λ) is negative for all λ roots.

When τ = 0, the characteristic equation reduces

$$\Delta(\lambda, 0) \coloneqq \lambda^3 + (a_2 + b_2)\lambda^2 + (a_1 + b_1)\lambda + a_0 + b_0 \qquad (16)$$

To determine the stability in this case we use the Routh-Hurwitz criterion.

This approach in its general form states that if a system is of order n, the polynomial can be taken in the form:

$$P(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_n = 0,$$

Where the coefficients $a_i,\ i = 0,1,\dots,n$ are all real.

Assuming aj = 0 for j> n, we require conditions on the coefficients such that the zeros of P (λ) have R (λ) <0. The necessary and sufficient conditions for this to be true are:

$$D_1 \coloneqq a_1 > 0, \quad D_2 \coloneqq \begin{vmatrix} a_1 & a_3 \\ 1 & a_2 \end{vmatrix} > 0, \quad D_3 \coloneqq \begin{vmatrix} a_1 & a_3 & a_5 \\ 1 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix} > 0,$$

$$D_k \coloneqq \begin{vmatrix} a_1 & a_3 & \cdot & \cdot & \cdot & \cdot \\ 1 & a_2 & a_4 & \cdot & \cdot & \cdot \\ 0 & a_1 & a_3 & \cdot & \cdot & \cdot \\ 0 & 1 & a_2 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & a_k \end{vmatrix} > 0 \quad , k = 1,2,\dots,n$$

For the case of a cubic equation

$$\lambda + c_1\lambda^2 + c_2\lambda + c_3 = 0 \qquad (17)$$

With real coefficients $c_1,\ c_2, c_3$ the conditions to $\Re(\lambda) < 0$ are:
$c_1 > 0,\ c_3 > 0,\ c_1c_2 - c_3 > 0.$

Thus in our case the balance (Y^*, K^*, R^*) is locally asymptotically stable if and only if

$$a_1 + b_1 > 0, a_0 + b_0 > 0 \quad \text{y} \quad (a_2 + b_2)(a_1 + b_1) - (a_0 + b_0) > 0 \qquad (18)$$

For τ> 0 case we will determine conditions so that λ = iω is a root of the characteristic equation, ie, that present the possibility of a Hopf bifurcation [6]. It replaces the previous value of λ is the characteristic equation (15) we have:

$$\begin{aligned}\Delta(i\omega, \tau) &= -\omega^3 i - a_2\omega^2 + a_1\omega i + a_0 \\ &\quad + (-b_2\omega^2 + b_1\omega i + b_0)e^{-i\omega\tau} \\ &= -\omega^3 i - a_2\omega^2 + a_1\omega i + a_0 \\ &\quad + (-b_2\omega^2 + b_1\omega i \\ &\quad + b_0)(\cos(\omega\tau) - isen\ (\omega\tau))\end{aligned}$$

And separating the above equation we obtain real and imaginary part, respectively, the following equations:

$$-\omega^2 a_2 + a_0 = (\omega^2 b_2 - b_0)\cos(\omega\tau) - b_1\omega sen\ (\omega\tau) \qquad (19)$$

$$-\omega^3 + a_1\omega = -b_1\omega\cos(\omega\tau) + (-\omega^2 b_2 + b_0)sen(\omega\tau) \quad (20)$$

To remove the trigonometric functions we raise both sides of (19) and (20) the square and add, after algebraic simplifications we obtain the following equation for ω:

$$\omega^6 + A_1\omega^4 + B_1\omega^2 + C_1 = 0, \quad (21)$$

Where

$$A_1 = a_2^2 - 2a_1 - b_2^2,$$
$$B_1 = a_1^2 - 2a_0a_2 + 2b_0b_2 - b_1^2,$$
$$C_1 = a_0^2 - b_0^2$$

Taking $z = \omega^2$ we can write the above equation as an equation of third degree, $z^3 + A_1z^2 + B_1z + C_1 = 0$.

Thus to find a root λ = iω seek the positive characteristic equation from the above equation solutions so that $\omega = \sqrt{z}$. Note too $\lambda = -i\omega$, the conjugate of the anterior root is a root of the characteristic equation.

Lemma 1.

Considering $D = A_1^2B_1^2 - 4B_1^3 - 4A_1^3C_1 - 27C_1^2 + 18A_1B_1C_1$, the discriminant of the above equation, the roots of equation (21) satisfy:

	Number of positive solutions	Conditions
(i)	0	$A_1 > 0, B_1 > 0, C_1 > 0, A_1B_1 - C_1 > 0$
		$C_1 < 0$, y ($B_1 \leq 0$ o $A_1 \geq 0$) o
(ii)	1	$C_1 < 0, B_1 > 0, A_1 < 0, D_1 > 0$ o
		$C_1 = 0, B_1 < 0$ o
		$C_1 = 0, B_1 = 0, A_1 < 0$
(iii)	2	$C_1 > 0, B_1 < 0$ o
		$A_1 < 0, D_1 \leq 0$ o
		$C_1 = 0, B_1 > 0, A_1 < 0$
(iv)	3	$C_1 < 0, B_1 > 0, A_1 < 0, D_1 \leq 0$

In case (i) when there are no positive solutions of the cubic equation, then the equation (21) can not have a real root $z = \sqrt{\omega}$, so by Rouche's theorem [6] the number of roots with positive real part is unchanged. Thus in this case the solution is stable equilibrium when (18) is satisfied.

In cases (ii) - (iv) can be found $\omega = \sqrt{z}$ real, in a way that $\lambda = \pm i\omega$ root of the characteristic equation (15), then you retaking the system (19) - (20) we can multiply equation (19) by $b_0 - \omega^2 b_2$, (20) by $b_1\omega^2$ adding the equations and eliminate the trigonometric function sin (ωτ) and solve cos (ωτ) obtaining:

$$\cos(\omega\tau) = \frac{\omega^4(b_1 - a_2b_2) + \omega^2(a_2b_0 + a_0b_2 - a_1b_1) - a_0b_0}{(b_1\omega)^2 + (b_0 - b_2\omega^2)^2}, \quad (22)$$

Whereas ω ^ * as given by Lemma 1 root clearance τ we obtain the value of parameter τ where a pair of conjugate pure imaginary roots occur:

$$\tau^* = \frac{1}{\omega^*}\arccos\left(\frac{(\omega^*)^4(b_1 - a_2b_2) + (\omega^*)^2(a_2b_0 + a_0b_2 - a_1b_1) - a_0b_0}{(b_1\omega)^2 + (b_0 - b_2(\omega^*)^2)^2}\right). \quad (23)$$

There is more than one value of τ ^ * due to the periodicity of the cosine function and the possible existence of 2 or 3 values ω ^ * (cases (iii) and (iv) of Lemma 1). Of these values τ ^ * take the smallest positive value. For other values of τ ^ * there may be other exchange of stability depending on the direction in which it again crosses the imaginary axis in each of these values of τ ^ *. Note that in the above expression is first necessary to set the parameters of the system (11), except for τ, to have ω ^ * ^ * and τ associated to the pair of imaginary roots. Henceforth τ ^ * be the smallest positive value given by (23) and ω ^ * root (21) from which it comes.

Now, to establish the Hopf bifurcation at τ = τ ^ *, we need to show that the imaginary axis from left to right cross. To do this it is determined whether $d\Re(\lambda(\tau))/d\tau|_{\tau=\tau^*} > 0$, that is, the derivative of the real part of the root λ (τ) of the characteristic equation in τ ^ * evaluated is positive. Then (15) Differentiating with respect to τ we have:

$$(3\lambda^2 + 2a_2\lambda + a_1)\frac{d\lambda}{d\tau} + \left(-\tau e^{-\lambda\tau}(b_2\lambda^2 + b_1\lambda + b_0)\right)\frac{d\lambda}{d\tau} - \lambda e^{-\lambda\tau}(b_2\lambda^2 + b_1\lambda + b_0) + e^{-\lambda\tau}(2b_2\lambda + b_1)\frac{d\lambda}{d\tau} = 0$$

Clearing $d\lambda/d\tau$ we have $\frac{d\lambda}{d\tau} = \frac{\lambda e^{-\lambda\tau}(b_2\lambda^2+b_1\lambda+b_0)}{3\lambda^2+2a_2\lambda+a_1-\tau e^{-\lambda\tau}(b_2\lambda^2+b_1\lambda+b_0)+e^{-\lambda\tau}(2b_2\lambda+b_1)}$ esto da:

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{3\lambda^2 + 2a_2\lambda + a_1 - \tau e^{-\lambda\tau}(b_2\lambda^2 + b_1\lambda + b_0) + e^{-\lambda\tau}(2b_2\lambda + b_1)}{\lambda e^{-\lambda\tau}(b_2\lambda^2 + b_1\lambda + b_0)}$$

$$= \frac{3\lambda^2 + 2a_2\lambda + a_1 + e^{-\lambda\tau}(2b_2\lambda + b_1)}{\lambda e^{-\lambda\tau}(b_2\lambda^2 + b_1\lambda + b_0)} - \frac{\tau}{\lambda}$$

$$= \frac{3\lambda^3 + 2a_2\lambda^2 + a_1\lambda + e^{-\lambda\tau}(2b_2\lambda^2 + b_1\lambda)}{\lambda^2 e^{-\lambda\tau}(b_2\lambda^2 + b_1\lambda + b_0)} - \frac{\tau}{\lambda}$$

$$= \frac{2\lambda^3 + a_2\lambda^2 - e^{-\lambda\tau}b_2\lambda^2 + \lambda^3 + a_2\lambda^2 + a_1\lambda - e^{-\lambda\tau}(b_2\lambda^2 + b_1\lambda)}{\lambda^2 e^{-\lambda\tau}(b_2\lambda^2 + b_1\lambda + b_0)} - \frac{\tau}{\lambda}$$

$$= \frac{2\lambda^3 + a_2\lambda^2 - e^{-\lambda\tau}b_2\lambda^2 - a_0 - e^{-\lambda\tau}b_0}{\lambda^2 e^{-\lambda\tau}(b_2\lambda^2 + b_1\lambda + b_0)} - \frac{\tau}{\lambda}$$

$$= \frac{2\lambda^3 + a_2\lambda^2 - a_0}{-\lambda^2(\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0)} + \frac{b_2\lambda^2 - b_0}{\lambda^2(b_2\lambda^2 + b_1\lambda + b_0)} - \frac{\tau}{\lambda}$$

Note that we use the equation (15) in various equalities. Now we find the sign or direction of the real part, equation (15) back to help simplify some expressions:

$$signo\left\{\frac{d\Re(\lambda(\tau))}{d\tau}\right\}\Bigg|_{\lambda=i\omega^*} = signo\left\{\Re\left(\frac{d\lambda}{d\tau}\right)^{-1}\right\}\Bigg|_{\lambda=i\omega^*}$$

$$= signo\left\{\Re\left[\frac{2\lambda^3 + a_2\lambda^2 - a_0}{-\lambda^2(\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0)}\right]_{\lambda=i\omega^*} + \Re\left[\frac{b_2\lambda^2 - b_0}{\lambda^2(b_2\lambda^2 + b_1\lambda + b_0)}\right]_{\lambda=i\omega^*} - \Re\left[\frac{\tau}{\lambda}\right]_{\lambda=i\omega^*}\right\}$$

$$= signo\left\{\Re\left[\frac{-2i(\omega^*)^3 - a_2(\omega^*)^2 - a_0}{(\omega^*)^2(-\omega^3 i - a_2(\omega^*)^2 + a_1\omega^* i + a_0)}\right] + \Re\left[\frac{-b_2(\omega^*)^2 - b_0}{-(\omega^*)^2(-b_2(\omega^*)^2 + b_1\omega^* i + b_0)}\right]\right\}$$

In the next step we consider the real part of the above equality, note that τ / λ is purely imaginary, obtaining:

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$$= signo\left\{\frac{2(\omega^*)^6 + (a_2^2 - 2a_1)(\omega^*)^4 - a_0^2}{(\omega^*)^2[(a_2(\omega^*)^2 - a_0)^2 + ((\omega^*)^3 - a_1\omega^*)^2]} + \frac{b_0^2 - b_2^2(\omega^*)^4}{(\omega^*)^2[(b_0 - b_2(\omega^*)^2)^2 + (b_1\omega^*)^2]}\right\}$$

The denominators of the above fractions are equal, this can be seen in the system (19) - (20) which can raise both sides to the square and we add equality mentioned:

$$= signo\left\{\frac{2(\omega^*)^6 + (a_2^2 - 2a_1 - b_2^2)(\omega^*)^4 + b_0^2 - a_0^2}{(\omega^*)^2[(a_2(\omega^*)^2 - a_0)^2 + ((\omega^*)^3 - a_1\omega^*)^2]}\right\}$$

Substituting

$b_0^2 - a_0^2$ the equation (21) and simplifying we have:

$$= signo\left\{\frac{3(\omega^*)^4 + 2(a_2^2 - 2a_1 - b_2^2)(\omega^*)^2 + (a_1^2 - 2a_2a_0 + 2b_2b_0 - b_1^2)}{[(a_2(\omega^*)^2 - a_0)^2 + ((\omega^*)^3 - a_1\omega^*)^2]}\right\}$$

$(a_2(\omega^*)^2 - a_0)^2 + ((\omega^*)^3 - a_1\omega^*)^2$ is positive, a Hopf bifurcation if present:

Based on the above we can establish the following:

$$3(\omega^*)^4 + 2(a_2^2 - 2a_1 - b_2^2)(\omega^*)^2 + a_1^2 - 2a_2a_0 + 2b_0b_2 - b_1^2 > 0. \quad (24)$$

Theorem 1

If condition (18), the equilibrium point (Y ^ *, K ^ *, R *) is satisfied is locally asymptotically stable when τ = 0. So is whether further fulfills the first condition of Lemma 1 when τ> 0.

Theorem 2

Suppose that satisfies (18) and one of the last three conditions of Lemma 1, as ω ^ * ^ * roots and τ the delay value associated with this result given in (23). Intone a Hopf bifurcation occurs if (24) is satisface.

Observation: Also you can have a Hopf bifurcation at $\tau = \tau^*$ when the characteristic equation $\tau = 0$ has exactly 2 roots with positive real part (equilibrium point (Y ^ *, K ^ *, R *) unstable) and $d\Re(\lambda(\tau))/d\tau|_{\tau=\tau^*} < 0$, es decir, (24) and satisfy with "<" instead of ">".

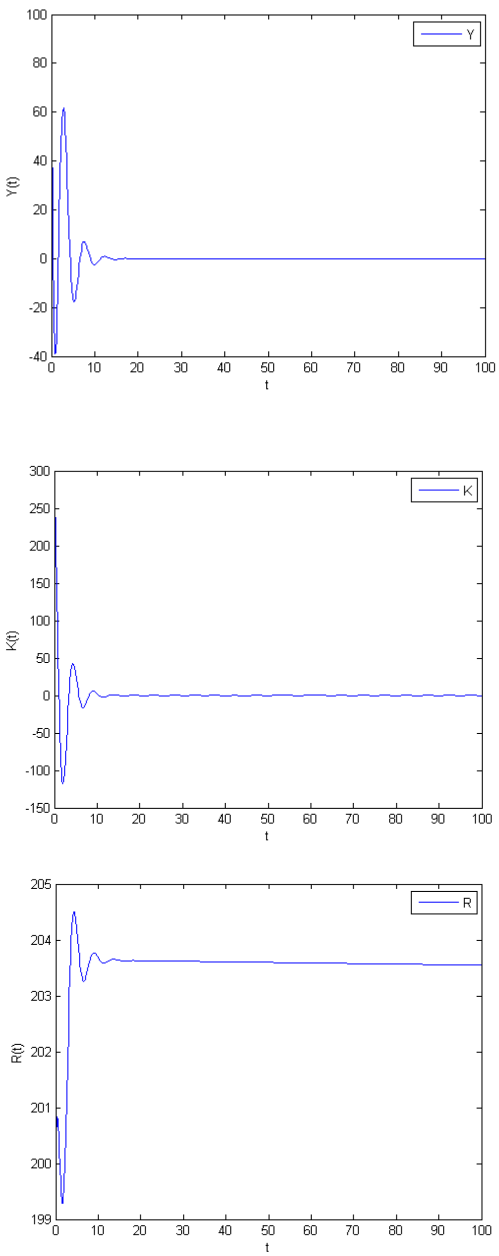
Numerical simulations

In this section we present some numerical simulations for certain fixed parameter values to illustrate the change in stability of the system (11) depending on the value of retardoτ. For our simulations we consider the following values for the parameters:

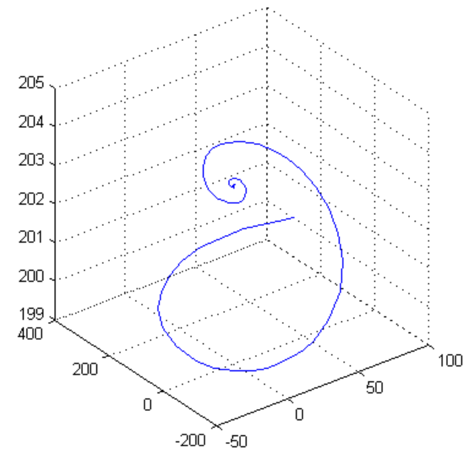
Parameter	Value
α	1
A	0.01
a	1
b	0.5
c	0.9
s	1
δ	0.1
β	1
γ	0.5
h	0.2
$\bar{R}$	0.0001
M	200

To consider the delay values τ = 1.3 to illustrate the stability and instability first value with the second.

The above graphs illustrate the stability of the system (11), taking as delay value $\tau = 1$.

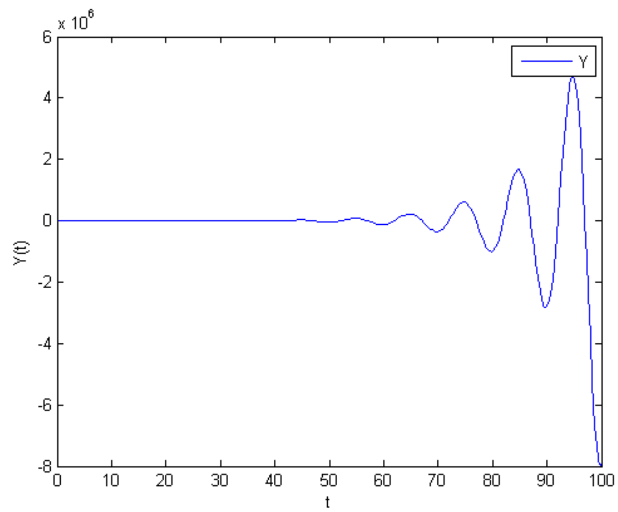


Graphic 1



The first three illustrate the dynamics of Y (t), K (t) and R (t), and the latter reflects the type attractor behavior of the equilibrium point.

The above graphs illustrate the loss of stability of the system (11), taking as delay value $\tau = 3$.



Graphic 2

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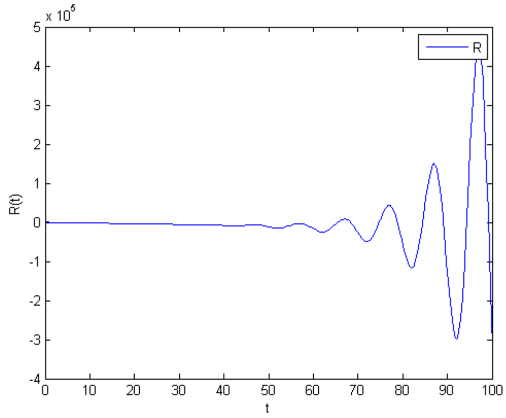

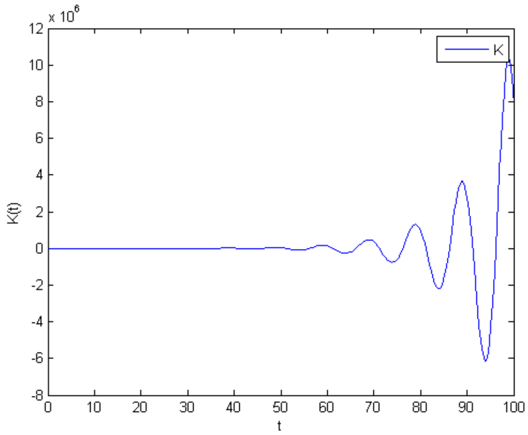

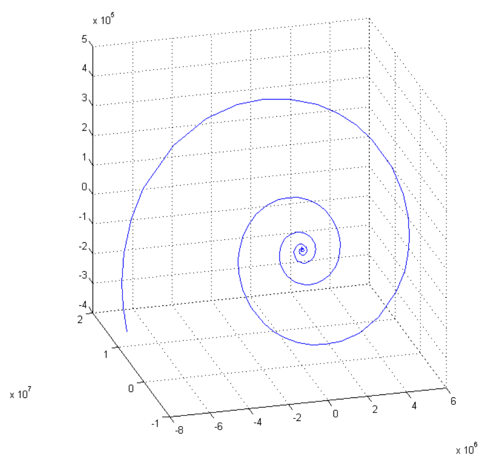

The first three illustrate the dynamics of Y (t), K (t) and R (t), and the latter reflects the behavior type repeller breakeven.

Conclusions

In this paper we focus on the analysis of mathematical way, we established conditions to ensure stability of the equilibrium change according to the delay and the occurrence of a Hopf bifurcation. In the numerical simulations we can see the loss of stability of the equilibrium to increase the value of retard possible future work can study the model

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